

Basis states for gravitons in non-perturbative loop representation space

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Abstract

A relation between the non-perturbative loop representation space (for use of exact quantization of general relativity) and the semi-classical loop representation space (for use of approximate quantization of general relativity with a flat background spacetime) is studied. A sector of (approximate) states and a sector of operators in the non-perturbative loop representation space are made related respectively to the physical states and the basic variables of the semi-classical loop representation space through a transformation. This transformation makes a construction of graviton states within the non-perturbative theory possible although the notion of gravitons originally emerged from semi-classical perturbative treatments. This transformation is “exact” in the sense that it does not contain an error term, a fact contrast to the previous construction of a similar transformation. This fact allows the interpretation that these graviton states represent free gravitons even after introducing the self-interaction of gravitons into the theory; the presence of an error term of order of possible perturbative self-interaction would spoil this interpretation. The existence of such an “exact” relation of the two theories supports the potential ability of the non-perturbative loop representation quantum gravity to address the physics of gravitons, namely quanta for large scale small fluctuations of gravitational field from the flat background spacetime.

1 Introduction

In order to quantize general relativity [1], the so-called loop representation [2, 3] has been applied both non-perturbatively (exactly) [3] and semi-classically (approximately) [4]. The latter is an approximation of the former and the physics described by the latter must be contained in the former if they are consistently related to each other. However, in order to quantize semi-classically, one first linearizes general

relativity at the classical level and then quantizes it. The resulting theory is mathematically different from the non-perturbatively quantized theory of exact general relativity. To find the mathematical structure of the semi-classical theory within the non-perturbative theory is highly non-trivial task. This seems a necessary step in order to say: this non-perturbative theory is really an “exact” physics theory and this semi-classical theory is really an “approximate” physics theory. We study a possible way of relating the two theories.

This study was initiated a few years ago [5, 6]. (An outline of the work is mentioned in section 5.) However, it was thought that the relation between the two theories was just an approximation useful for “estimating” the approximate free graviton states and that the inclusion of self-interaction spoiled the approximation. In order to include the self-interaction effects, one had to wait for the development of computation techniques for non-perturbative transition amplitudes such as regularizations of the exact Hamiltonian constraint [7, 8, 9] and sum-over-surfaces formulations [10, 11, 12, 13, 14, 15].

Recently [16], it was re-examined and speculated that there might exist an “exact” relation between the two theories when considered only at the vicinity of the flat spacetime. (The “exact” does not mean the two theories have the same mathematical structure. This “exactness” should be understood in a similar sense that any space-time metric $g_{\mu\nu}$ can be written as the sum of the flat spacetime metric and a deviation from it, namely $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. This can be done always and exactly and this fact itself does not mean that this splitting is an approximation. However, this splitting of metric is physically sensible as an approximation only if $h_{\mu\nu}$ is small enough compared to unity.) If it is true, then not only can the free graviton states determined in the non-perturbative theory be kept meaningful after the inclusion of self-interaction but also may the relation between the two theories allow to include the self-interaction effects “perturbatively” within the non-perturbative representation space [17].

We study in this paper such an “exact” relation of the two theories. We present a closed form of the relation of states of the two theories without error terms. It was previously thought that error terms were unavoidable and would make the inclusion of self-interaction terms in the Hamiltonian obscure. Although the relation has a closed form, still it is an approximation meaningful at the vicinity of the flat spacetime. Possible applications of the relation such as computations of graviton-graviton scattering amplitudes would be understood as approximation schemes as a tentative step toward the developments of non-perturbative computation techniques (see [18] for recent review in this direction).

Some of the important consequences of the relation between the non-perturbative and semi-classical theories are the following. The semi-classical theory has a background flat spacetime (a classical solution) but the non-perturbative theory does not. The relation we study requires the presence of a particular spin-network, called “weave” [19, 20], in the non-perturbative representation space. The weave presents a discrete Planck scale structure but approximates the flat space when probed at scales far larger than the Planck length. In a sense the weave represents a classical limit of the theory. Because of the presence of the weave, we can show the existence of an

explicit relation of the two theories meaning that the non-perturbative theory contains the graviton physics described by the semi-classical theory. In other words, the existence of the relation of the two theories supports the existence of a possible discrete Planck scale structure of space presented by the weave without conflictions with large scale continuum pictures. The Planck scale corrections due to the discrete structure of the weave is absorbed to the Planck length constant to redefine a “coarse-grained” constant and hence the Planck scale structure of the weave cannot be seen from the semi-classical theory.

The graviton states determined within the non-perturbative theory are quantum states dressed with virtual loop gravitons around the weave and can be seen as quantum states of the semi-classical theory dressed with virtual gravitons around the flat space when probed at large scales. They are supposed to contain dynamical information, although it is an approximation sensible at the vicinity of flat spacetime, and hence they together with possible graviton-graviton scattering amplitudes between them represent the flat spacetime with large scale small fluctuations of gravitational field, namely gravitons.

This is a physical picture emerging from the non-perturbative loop representation space in addition to other physical aspects [21, 22, 23, 24, 25, 26].

Illustration of the idea

In the rest of the introduction section we illustrate the idea we follow in terms of a simple model; namely quantum mechanics of a non-relativistic particle in one-dimensional space. It is then followed by an outline of the following sections, where we make our discussions parallel to the illustration.

The Hamiltonian of a particle with a mass m at position X in a potential $V(X)$ is given by $H := \frac{P^2}{2m} + V(X)$. Here, P is the momentum of the particle and is an operator satisfying the commutation relation $[X, P] = i\hbar$ in a representation space. We work in a representation in which X is a multiplication number and states are function of X .

In general, to find eigen states of the Hamiltonian is difficult and one needs an approximation in order to discuss physics. Let us suppose that the potential has a minimum at the origin $X = 0$ and varies slowly at the vicinity of the origin. We specify the vicinity by the range $-\varepsilon < X < \varepsilon$ with a positive small real number ε . Rewrite the potential as an expansion about the origin $V(X) = V(0) + \frac{1}{2}V''(0)X^2 + \mathcal{O}(\varepsilon^3)$, and define a Hamiltonian approximating the exact Hamiltonian as $H' := \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$. Here, ω is a constant defined such that $m\omega^2 = V''(0)$ and x is the position of the particle relative to the position of the minimum of the potential, namely $x := X$. ($x = X$ exactly holds always, but the approximate model is physically sensible only when x is in the range.) The momentum p is an operator satisfying the commutation relation $[x, p] = i\hbar$ in a representation space. We work in a representation in which x is a multiplication number and states are functions of x . In general, the two representation spaces are different and we have to know their relation in order to

discuss physics approximating some aspect of the exact model.

Given a state of the exact model $f(X)$, define a transformation $\exp[\frac{i}{\hbar}x \cdot P]$. When applied to the state $f(X)$, this transformation produces a state $f(x)$ of the approximate model if evaluated at the origin $X = 0$. This is the relation of the states in the two representation spaces. The function of the corresponding state of the approximate model is the same as the function of the exact model with X replaced by x . But we know $x = X$ by definition and hence the corresponding states in the two models are exactly identical. This seems a trivial result. But if we go to quantum gravity, which is a field theory and has an infinite number of degrees of freedom, then an analogous consideration gives non-trivial consequences.

Since the approximate model can be solved exactly, a solution $\psi(x)$ can be understood as one corresponding to a state $\psi(X)$ of the exact model and the latter can be interpreted as an approximate solution of the exact model. Its physical meaning is the same as the physical meaning of $\psi(x)$ of the approximate model and this physics can be understood as contained in the exact model.

In the following sections we discuss a relation between the non-perturbative and semi-classical theories of loop quantum gravity. (The relation is an analogue of the relation between the two models in the illustration.) In section 2 we define basis states and functions of them in the non-perturbative theory (the analogues of X and $f(X)$ of the exact model in the illustration). In section 3 we review the basis states and functions of them of the semi-classical theory (the analogues of x and $f(x)$ of the approximate model in the illustration). In section 4 we define a transformation from one basis to the other of the two theories (the analogue of $\exp[\frac{i}{\hbar}x \cdot P]$ in the illustration). Then show that the analytic function of the basis states in the non-perturbative theory is transformed to the same analytic function of the basis states of the semi-classical theory if evaluated at the weave corresponding to flat space. (This is analogous to the fact that $f(x)$ is transformed to $f(X)$ through $\exp[\frac{i}{\hbar}x \cdot P]$ if evaluated at the origin $X = 0$ in the illustration.) The transformation can be defined at any spin-network “exactly” without error terms but it is physically sensible as an approximation when it is evaluated at the weave. (This situation is analogous to the fact that $\exp[\frac{i}{\hbar}x \cdot P]$ can be defined at any position X exactly without error terms but it is physically sensible as an approximation when it is evaluated at the origin $X = 0$.) We compare the present work with the previous work in section 5. We conclude our study at the end.

2 Basis states in the non-perturbative theory

The states in the non-perturbative theory are functions on spin-networks Γ , denoted by $\Psi(\Gamma) = \langle \Gamma | \Psi \rangle$ [27].

We restrict ourselves to a family of states, which serve as our domain states for gravitons, as follows.

$$\langle \Gamma | f(G^\pm) \rangle := N^{-1}(\Gamma) \sum_{\{\gamma\} \in \Gamma} (-2)^{n(\{\gamma\})-1} (-1)^{c(\{\gamma\})} f(G^\pm[\{\gamma\}]). \quad (1)$$

Here $\{\gamma\}$ is a set of single loops made out of all the segments belonging to the edges of Γ and the $n(\{\gamma\})$ is the number of single loops in $\{\gamma\}$. From a given spin-network Γ , a finite number of sets of single loops can be constructed as follows. Place $2j$ identical segments (or lines) on an edge with spin- j . Repeat for the other edges of Γ . By connecting every two segments (not on the same edge) meeting at a vertex, fix a set of single loops out of all the segments. Permutate the connectivity of two segments on one edge of Γ . In other words, choose two segments on a single edge and cut them at the middle, then reconnect them in the alternative way so that any of the resulting segments does not retrace its way. This permutation creates another set of single loops. Repeat permutations of the connectivity of the segments on the edge in all the possible ways. The number of the permutations of segments performed on a single edge with spin- j is $(2j)!$. Do the same thing for all the edges. This procedure creates all the possible sets of single loops out of the segments belonging to the edges of Γ . $\{\gamma\} \in \Gamma$ means one of the sets defined from Γ and $c(\{\gamma\})$ is the number of the permutations performed to define $\{\gamma\}$. $N(\Gamma)$ is the number of all the possible permutations for Γ , $N(\Gamma) := \prod_{e \in \Gamma} (2j_e)!$; $e \in \Gamma$ means an edge e in Γ and $2j_e$ is the number of the segments placed on the edge e (j_e is the spin of the edge e). If Γ has a vertex with valence more than three, we understand that the vertex is made of a trivalent spin-network with “virtual” edges [27] representing an intertwiner consistent with the adjacent edges to the vertex. A virtual edge is an edge without (parameter) length.

$f(G)$ is an analytic function of G and $f(G^\pm[\{\gamma\}])$ has the form

$$f(G^\pm[\{\gamma\}]) := \sum_{n=0}^{\infty} l_p^n \int d^3x_1 \cdots d^3x_n f_{\sigma_1 \cdots \sigma_n}(x_1 \cdots x_n) G^{\sigma_1}[x_1, \{\gamma\}] \cdots G^{\sigma_n}[x_n, \{\gamma\}], \quad (2)$$

$$G^\pm[x, \{\gamma\}] := \sum_{\gamma \in \{\gamma\}} G^\pm[x, \gamma]. \quad (3)$$

Here $f_{\sigma_1 \cdots \sigma_n}(x_1 \cdots x_n)$ is a coefficient (a function of space points) subject to some condition. This condition is to be determined by the transformation discussed in section 4. σ_i ($i = 1, \dots, n$) are summed over $+$ and $-$. l_p is the Planck length. $\gamma \in \{\gamma\}$ means a single loop in $\{\gamma\}$. $G^\pm[x, \gamma]$ for a single loop γ is to be defined below. (These states are analogues of $f(X)$ of the exact model in the illustration.) If Γ is just a single loop γ , then $\langle \gamma | f(G^\pm) \rangle = f(G^\pm[\gamma])$ and in particular we denote $\langle \gamma | G^\pm(x) \rangle = G^\pm[x, \gamma]$ our basis states for gravitons. (They are analogues of X of the exact model in the illustration.)

Define operators $\hat{H}^\pm(x)$ in the space spanned by the family of states as follows.

$$\begin{aligned} \langle \Gamma | \hat{H}^\pm(x) | f(G^\pm) \rangle &:= N^{-1}(\Gamma) \sum_{\{\gamma\} \in \Gamma} (-2)^{n(\{\gamma\})-1} (-1)^{c(\{\gamma\})} \times \\ &\quad (-l_p^2) \int d^3y f_r(x, y) G^\pm[y, \{\gamma\}] f(G^\pm[\{\gamma\}]). \end{aligned} \quad (4)$$

Here, $f_r(x, y)$ is a function with a scale parameter r . The precise definitions of this function and the scale r are to be determined by the transformation discussed in the

section 4. (Although these operators are not diagonal, they are analogues of X as an operator of the exact model in the illustration.)

Suppose Γ is a spin-network and η is a single (parameterized) loop without self-intersection (a special case of the spin-network). Then $\Gamma \cup \eta$ is another spin-network consisting of Γ and η if they share at most countable number of points (intersections between them) creating new vertices. If they intersect with each other, then the vertex corresponding to one of the intersections is labeled by the singlet intertwiner (in other words, the trivial intertwiner or spin-0 intertwiner). Also $\Gamma\eta - \Gamma\eta^{-1}$ is another spin-network consisting of Γ and η if they share at most countable number of points (intersections between them). In this case, they intersect at least once with each other. The notation indicates that η is attached on Γ at one of the intersections in the two different ways and the two terms are summed with appropriate signs. In spin-network language, the vertex corresponding to this intersection is labeled by one of the triplet intertwiners (or spin-1 intertwiners) and the vertices corresponding to the other intersections are labeled by the singlet intertwiner if any. Similarly, $\Gamma\eta\chi - \Gamma\eta^{-1}\chi - \Gamma\eta\chi^{-1} + \Gamma\eta^{-1}\chi^{-1}$ is a spin-network. Γ shares a vertex having one of the triplet intertwiners with η and χ in each.

From the definition of the domain states for gravitons, the values of the state for the spin-networks $\Gamma \cup \eta$, $\Gamma\eta - \Gamma\eta^{-1}$ and $\Gamma\eta\chi - \Gamma\eta^{-1}\chi - \Gamma\eta\chi^{-1} + \Gamma\eta^{-1}\chi^{-1}$ are determined as follows.

$$\langle \Gamma \cup \eta | f(G^\pm) \rangle = N^{-1}(\Gamma) \sum_{\{\gamma\} \in \Gamma} (-2)^{n(\{\gamma\} \cup \eta) - 1} (-1)^{c(\{\gamma\})} f(G^\pm[\{\gamma\} \cup \eta]), \quad (5)$$

$$\langle \Gamma_e \eta - \Gamma_e \eta^{-1} | f(G^\pm) \rangle = N^{-1}(\Gamma) \sum_{\{\gamma\} \in \Gamma} (-2)^{n(\{\gamma\}) - 1} (-1)^{c(\{\gamma\})} \times$$

$$[(2j_e)2!]^{-1} \sum_{i \in e} [f(G^\pm[\{\gamma\}_i \eta]) - f(G^\pm[\{\gamma\}_i \eta^{-1}])], \quad (6)$$

$$\begin{aligned} \langle \Gamma_{ee'} \eta \chi - \Gamma_{ee'} \eta^{-1} \chi - \Gamma_{ee'} \eta \chi^{-1} + \Gamma_{ee'} \eta^{-1} \chi^{-1} | f(G^\pm) \rangle = \\ N^{-1}(\Gamma) \sum_{\{\gamma\} \in \Gamma} (-2)^{n(\{\gamma\}) - 1} (-1)^{c(\{\gamma\})} [(2j_e)(2j_{e'})(2!)^2]^{-1} \times \\ \sum_{i \in e} \sum_{j \in e'} [f(G^\pm[\{\gamma\}_{ij} \eta \chi]) - f(G^\pm[\{\gamma\}_{ij} \eta^{-1} \chi]) \\ - f(G^\pm[\{\gamma\}_{ij} \eta \chi^{-1}]) + f(G^\pm[\{\gamma\}_{ij} \eta^{-1} \chi^{-1}])] . \end{aligned} \quad (7)$$

Here e and e' are the edges of Γ to which η and χ are attached and i and j are one of the segments belonging to the edges e and e' respectively. $\{\gamma\}_i \eta$ means η is attached to the segment i and hence one of single loops in $\{\gamma\}$ to which the segment i belongs is modified while the other loops in $\{\gamma\}$ stay the same.

One of the basic variables of the non-perturbative theory, called loop variables, is

$$T[\Gamma] := N^{-1}(\Gamma) \sum_{\{\gamma\} \in \Gamma} (-1)^{n(\{\gamma\})} (-1)^{c(\{\gamma\})} \prod_{\gamma \in \{\gamma\}} \text{Tr}[\gamma]. \quad (8)$$

$\text{Tr}[\gamma]$ is the trace of the holonomy of the Ashtekar connection variable along the loop γ . Note that $\text{Tr}[\gamma]$ goes to 2 if γ shrinks to a point. If Γ is just a single loop η , then

the corresponding variable is $T[\eta] = -\text{Tr}[\eta]$. Its action as an operator on the state is

$$\langle \Gamma | \hat{T}[\eta] := \langle \Gamma \cup \eta |. \quad (9)$$

Because of the definition of the action, the states satisfy the following conditions consistently with the conventions adopted in the definition of $T[\Gamma]$.

$$\langle \Gamma \cup \eta | = \langle \Gamma^{-1} \cup \eta | = \langle \Gamma \cup \eta^{-1} | \quad (10)$$

$$\langle \Gamma \eta - \Gamma \eta^{-1} | = -\langle \Gamma \eta^{-1} - \Gamma \eta | \quad (11)$$

$$\lim_{\eta \rightarrow \cdot} \langle \Gamma \cup \eta | = -2\langle \Gamma | \quad (12)$$

The conventions used in this paper are similar to those adapted for tangle theoretic recoupling theory technology [28, 29] and the difference does not make problems in order to understand the main ideas of this study. The point of the conventions is such that the loop operators are defined locally and do not depend on how the spin-network under consideration is decomposed to sets of single loops. Accordingly such non-local information of spin-networks is hidden in the states. Since our task is to define a family of states explicitly, our states have to contain such information.

Before defining other operators and $G^\pm[x, \gamma]$, which contain line integrals along an edge or a loop, we introduce a regularization of parameterized loops. The loops used in the theory are orientation preserved reparameterization invariant. Divide a loop to N pieces of curves each of which has parameter length ρ . The pieces are labeled by $k = 1, 2, \dots, N$ in the order of the orientation of the loop. Each piece is assumed to be smooth. If the loop has a self-intersection, then we divide the loop such that two pieces contain the intersection point. If the loop intersects with another loop, then we divide the loop such that a piece of the loop shares the intersection point with one piece of the other loop. At the end of all the calculations, take the limits $\rho \rightarrow 0$ and $N \rightarrow \infty$ such that ρN is finite. We define the line integral along a loop γ as

$$\sum_{k=1}^N \rho \dot{\gamma}^a(k) \cdots \xrightarrow{\rho \rightarrow 0, N \rightarrow \infty} \oint dt \dot{\gamma}^a(t) \cdots \equiv \oint_\gamma dx^a \cdots \quad (13)$$

where $\dot{\gamma}^a(k)$ is the tangent vector at the k -th piece of the loop γ . We keep ρ and N finite until the end of all the calculations although ρ is “small” and N is “large”. The regularization for the line integral for an edge of the spin-network is defined similarly. In this case, the vertices at the ends of the edge are treated in the same way as intersections of the loops are treated.

The action of another loop operator we need is defined as follows.

$$\langle \Gamma | \hat{T}^a[\eta](\eta(s)) := -il_p^2 \sum_{e \in \Gamma} 2j_e \sum_{t=1}^N \rho \dot{l}_e^a(t) \delta^3(\eta(s), l_e(t)) \langle \Gamma_e \eta - \Gamma_e \eta^{-1} |. \quad (14)$$

In terms of this operator, define another operator

$$\hat{D}^{ab}(x, \delta) := -\frac{1}{2} \frac{1}{\pi \delta^2} \sum_{s=1}^N \hat{T}^b[\eta_{x,\delta}^a](\eta(s)), \quad (15)$$

where $\eta_{x,\delta}^a$ is a parameterized circle with radius δ centered at x and normal to the a -direction. This operator with the limit $\delta \rightarrow 0$ obeys the Leibnitz rule. (This operator, or more precisely $\lim_{\delta \rightarrow 0} \epsilon_{abc}(\partial^b/|\partial|^2)\hat{D}^{ci}(x, \delta)$, is an analogue of P of the exact model in the illustration.) In the single equation, we often simplify notations for loops such as $\eta \equiv \eta_x \equiv \eta_{x,\delta} \equiv \eta_{x,\delta}^a$ if it is clear from the context.

We define $G^\pm[x, \gamma]$ as follows. Here γ and η are single loops.

$$\begin{aligned} G^\pm[x, \gamma] &:= \frac{1}{2} \sum_l \nu_l \int d^3y \left| \sum_{k=1}^N \rho \dot{\gamma}^a(k) \delta^3(\gamma(k), y) \omega_a^{\pm l}(y - x) \right| \\ &= \frac{1}{2} \sum_l \nu_l \sum_{k=1}^N \left| \rho \dot{\gamma}^a(k) \omega_a^{\pm l}(\gamma(k) - x) \right|, \end{aligned} \quad (16)$$

$$\begin{aligned} G^\pm[x, \gamma\eta] &:= \frac{1}{2} \sum_l \nu_l \int d^3y \left| \sum_{k=1}^N \rho \dot{\gamma}^a(k) \delta^3(\gamma(k), y) \omega_a^{\pm l}(y - x) \right. \\ &\quad \left. + \sum_{k=1}^N \rho \dot{\eta}^b(k) \delta^3(\eta(k), y) \omega_b^{\pm l}(y - x) \right| \\ &= G^\pm[x, \gamma] + G^\pm[x, \eta] + \\ &\quad \frac{1}{2} \rho \sum_l \nu_l \left| \dot{\gamma}^a(t) \omega_a^{\pm l}(\gamma(t) - x) + \dot{\eta}^b(s) \omega_b^{\pm l}(\eta(s) - x) \right|. \end{aligned} \quad (17)$$

Here $\gamma\eta$ is a single loop consisting of γ and η with an intersection. (Note that $\gamma\eta$ is different from $\gamma \cup \eta$. The latter means two single loops γ and η .) The last term in the second equation is the contribution from the intersection and would vanish if one naively takes the limit $\rho \rightarrow 0$. However, we do not do so until all the calculations are finished. t and s are the labels of two pieces of curves containing the intersection.

$\omega_a^{\pm l}$ is a covector defined as follows.

$$\omega_a^{\pm l}(x) := \omega_a^l(\pm x) \quad (18)$$

$$\sum_{l=1}^3 \frac{1}{2} \nu_l \frac{\omega_a^l(x) \omega_b^l(x)}{\|\omega^l(x)\|} = I_{ab}(x) := (2\pi)^{-3} \int d^3k |k|^{-2} m_a(k) m_b(k) e^{ik \cdot x}. \quad (19)$$

It is known that I_{ab} exists and can be computed explicitly [6]. ω_a^l ($l = 1, 2, 3$) are the eigen vectors of I_{ab} and can be computed for some values of x . $\|\omega^l(x)\|$ is the norm of ω_a^l at x . $\frac{1}{2} \nu_l \|\omega^l\|$ ($l = 1, 2, 3$) are the eigen values of I_{ab} and ν_l ($l = 1, 2, 3$) are the signs of the eigen values.

If Γ and η are smooth at the point they share and η shrinks to the point, then $G^\pm[x, \gamma\eta]$ can be expanded as

$$\begin{aligned} G^\pm[x, \gamma\eta] &= G^\pm[x, \gamma] + G^\pm[x, \eta] + \\ &\quad \frac{1}{2} \rho \sum_l \nu_l \frac{\dot{\gamma}(t) \cdot \omega^{\pm l}(\gamma(t) - x)}{|\dot{\gamma}(t) \cdot \omega^{\pm l}(\gamma(t) - x)|} \dot{\eta}(s) \cdot \omega^{\pm l}(\eta(s) - x) + \dots \end{aligned} \quad (20)$$

Here, the dot products are simplifications of notation implying, for example, $\dot{\gamma} \cdot \omega := \dot{\gamma}^a \omega_a$.

$G^\pm[x, \gamma]$ shows the following properties. Let γ and γ' be single loops and attach other loops η and χ on them. Assume that the four loops are smooth at the points any two of them share. Then

$$G^\pm[x, \gamma\eta] - G^\pm[x, \gamma\eta^{-1}] = \rho \sum_l \nu_l \frac{\dot{\gamma}(t) \cdot \omega^{\pm l}(\gamma(t) - x)}{|\dot{\gamma}(t) \cdot \omega^{\pm l}(\gamma(t) - x)|} \dot{\eta}(s) \cdot \omega^{\pm l}(\eta(s) - x) + \mathcal{O}(|\dot{\eta}|^2), \quad (21)$$

$$G^\pm[x, \gamma\eta\chi] - G^\pm[x, \gamma\eta^{-1}\chi] - G^\pm[x, \gamma\eta\chi^{-1}] + G^\pm[x, \gamma\eta^{-1}\chi^{-1}] = 0 + \mathcal{O}(|\dot{\eta}|^2) + \mathcal{O}(|\dot{\chi}|^2), \quad (22)$$

$$G^\pm[x, \gamma\eta \cup \gamma'\chi] - G^\pm[x, \gamma\eta^{-1} \cup \gamma'\chi] - G^\pm[x, \gamma\eta \cup \gamma'\chi^{-1}] + G^\pm[x, \gamma\eta^{-1} \cup \gamma'\chi^{-1}] = 0 + \mathcal{O}(|\dot{\eta}|^2) + \mathcal{O}(|\dot{\chi}|^2). \quad (23)$$

in the limit that each of the attached loops η and χ shrinks to a point.

3 Basis states of the semi-classical theory

The semi-classical theory describes large scale small fluctuations of gravitational field from a flat background spacetime. The smallness of the field variables is specified by a dimensionless possitive real parameter $\varepsilon \ll 1$. Since the theory contains a scale parameter l_p , namely the Planck length, another scale $r := l_p/\varepsilon$ enters the theory. This scale parameter r defines the typical scale of fluctuations of gravitational field [5].

The basis states of the semi-classical theory are $F^\pm[x, \vec{\alpha}]$, functions of a triplet of loops $\vec{\alpha} := (\alpha^1, \alpha^2, \alpha^3)$, defined as the Fourier transform of $|k|^{-2}F^\pm[k, \vec{\alpha}]$ by

$$F^\pm[x, \vec{\alpha}] := (2\pi)^{-3/2} \int d^3k |k|^{-2} F^\pm[k, \vec{\alpha}] e^{ik \cdot x}. \quad (24)$$

Here $F^\pm[k, \vec{\alpha}]$ are the two symmetric traceless transverse components of $F^a[k, \alpha^i]$: namely

$$F^+[k, \vec{\alpha}] := F^a[k, \alpha^i] \bar{m}_a(k) \bar{m}_i(k), \quad (25)$$

$$F^-[k, \vec{\alpha}] := F^a[k, \alpha^i] m_a(k) m_i(k). \quad (26)$$

$F^a[k, \alpha^i]$ is the Fourier transform of $F^a[x, \alpha^i] := \oint_{\alpha^i} dy^a \delta^3(x - y)$ defined by

$$\begin{aligned} F^a[k, \alpha^i] &:= (2\pi)^{-3/2} \int d^3x F^a[x, \alpha^i] e^{-ik \cdot x} \\ &= (2\pi)^{-3/2} \oint ds (\dot{\alpha}^i)^a(s) e^{-ik \cdot \alpha^i(s)}. \end{aligned} \quad (27)$$

m_a and \bar{m}_a are polarization vectors satisfying

$$m_a(k) \bar{m}^a(k) = 1, \quad m_a(k) m^a(k) = m_a(k) k^a = 0, \quad (28)$$

$$\bar{m}_a(k) = -m_a(-k), \quad \epsilon_{abc} k^a m^b(k) \bar{m}^c(k) = -i|k|. \quad (29)$$

($F^\pm[x, \vec{\alpha}]$ are analogues of x of the approximate model in the illustration.)

In terms of the basis states, the domain states of the semi-classical theory are defined as follows.

$$\begin{aligned} \langle \vec{\alpha} | g(F^\pm) \rangle &:= g(F^\pm[\vec{\alpha}]) := \\ &\sum_{n=0}^{\infty} l_p^n \int d^3x_1 \cdots d^3x_n g_{\sigma_1 \dots \sigma_n}(x_1 \cdots x_n) F^{\sigma_1}[x_1, \vec{\alpha}] \cdots F^{\sigma_n}[x_n, \vec{\alpha}]. \end{aligned} \quad (30)$$

Here $g_{\sigma_1 \dots \sigma_n}(x_1 \cdots x_n)$ is a slowly varying function over the scale $r := l_p/\varepsilon$ with respect to any of its arguments. (These states are analogues of $f(x)$ of the approximate model in the illustration.) The actions of the basic operators $\hat{h}^\pm(x)$ and $\hat{B}^\pm(x)$ are defined on the states as follows.

$$\langle \vec{\alpha} | \hat{h}^\pm(x) | g(F^\pm) \rangle := -l_p^2 \int d^3y g_r(x-y) F^\pm[y, \vec{\alpha}] \langle \vec{\alpha} | g(F^\pm) \rangle \quad (31)$$

$$\langle \vec{\alpha} | \hat{B}^\pm(x) | g(F^\pm) \rangle := \pm \langle \vec{\alpha} | \frac{\delta g}{\delta F^\pm[x]} \rangle. \quad (32)$$

with $g_r(x) := (2\pi r^2)^{-3/2} \exp[-x^2/2r^2]$. (These operators are analogues of x and p as operators of the approximate model in the illustration.)

The semi-classical theory of loop quantum gravity was constructed [4] in terms of complex variables. As long as the self-interaction terms in the Hamiltonian are truncated, the linearized reality conditions are successfully incorporated to determine an inner product with respect to which the annihilation and creation operators a and a^\dagger satisfy the required commutation relation $[a, a^\dagger] = 1$.

However, if one tries to include the self-interaction terms perturbatively, then one has to take the higher order terms in the linearized reality conditions into account; otherwise, the definition of the Hamiltonian in terms of the linearized operators become obscure. This procedure spoils the required commutation relation since the annihilation and creation operators cannot be anymore linear in the basic linearized canonical variables in order for the inner product to incorporate the linearized reality conditions to higher order terms. Without well defined annihilation and creation operators, one cannot construct the Fock space description of gravitons.

This difficulty can be easily overcome if one uses real variables [30] instead of complex variables. In addition, the real variable formulation fixes chiral asymmetry of the complex variable formulation. Moreover, most of the technologies developed [4] for the semi-classical theory are still valid and the considerations presented in this paper can be applied to both complex and real variable formulations. These facts have been recently realized [17].

Although it is unclear at present how the complex variable formulation can overcome this difficulty, the complex variable formulation has some interesting features [31] and should be deserved for further investigations [32, 33].

4 Transformation

We define an operator in terms of $\hat{D}^{ab}(x, \delta)$ in the non-perturbative representation state space.

$$\hat{U}(\vec{\alpha}) := \lim_{\delta \rightarrow 0} \exp \left[-\frac{i}{2} \sum_i \int d^3x \epsilon_{abc} F_\xi^a[x, \alpha^i] \frac{\partial^b}{|\partial|^2} \hat{D}^{ci}(x, \delta) \right], \quad (33)$$

with $F_\xi^a[x, \alpha^i] := \int d^3y g_\xi(x - y) F^a[y, \alpha^i]$. Here, ξ is a parameter such that $l_p \ll \xi \ll r := l_p/\varepsilon$. (This operator is an analogue of $\exp[\frac{i}{\hbar} x \cdot P]$ in the illustration.)

Then define a transformation \mathcal{N} from the non-perturbative to the semi-classical state space in terms of $\hat{U}(\vec{\alpha})$ such that

$$\langle \vec{\alpha} | \mathcal{N} = \langle \Delta | \hat{U}(\vec{\alpha}). \quad (34)$$

This abstract notation of transformation means that the transformation \mathcal{N} brings a state $\Psi(\Gamma) = \langle \Gamma | \Psi \rangle$ in the non-perturbative theory to a state $\psi(\vec{\alpha}) = \langle \vec{\alpha} | \psi \rangle = \langle \vec{\alpha} | \mathcal{N} | \Psi \rangle = \langle \Delta | \hat{U}(\vec{\alpha}) | \Psi \rangle$ in the semi-classical theory. Here Δ is a spin-network called “weave” [19]. Δ approximates the flat space in the sense that if one computes a geometrical quantity at large scales such as the area of a macroscopic surface on the weave then its value coincides with the value of the same quantity computed in terms of the flat space metric up to a small error of order l_p/L . l_p is the Planck scale and L is the scale of the surface under consideration.

We apply $\hat{U}(\vec{\alpha})$ to the family of states, $|f(G^\pm)\rangle$, and evaluate it at the weave Δ . (The evaluation of $\hat{U}(\vec{\alpha})$ at the weave Δ representing the flat space is analogous to the evaluation of $\exp[\frac{i}{\hbar} x \cdot P]$ at the origin $X = 0$, around which the approximate model is physically sensible as an approximation of the exact model in the illustration.) We perform calculations step by step below.

First, apply the operator $\hat{D}^{ab}(x, \delta)$ to the state $|f(G^\pm)\rangle$ and evaluate it at the weave Δ and take the limit $\delta \rightarrow 0$.

$$\begin{aligned} \lim_{\delta \rightarrow 0} \langle \Delta | \hat{D}^{ab}(x, \delta) | f(G^\pm) \rangle &= N^{-1}(\Delta) \sum_{\{\gamma\} \in \Delta} (-2)^{n(\{\gamma\})-1} (-1)^{c(\{\gamma\})} \times \\ &\int d^3y \frac{\delta f(G^\pm[\{\gamma\}])}{\delta G^\sigma[y, \{\gamma\}]} \lim_{\delta \rightarrow 0} \left(\frac{1}{2} \hat{D}^{ab}(x, \delta) G^\sigma[y, \{\gamma\}] \right). \end{aligned} \quad (35)$$

Here we have used the fact that the operator $\hat{D}^{ab}(x, \delta)$ obeys the Leibnitz rule. Note that the emergence of the factor of $\frac{1}{2}$ in front of \hat{D}^{ab} is due to the fact that the operation of the operator on G^\pm produces two terms in each action to the segment on an edge.

Next, apply the operator \hat{D}^{ab} twice to the state $|f(G^\pm)\rangle$ and evaluate it at the weave Δ and then take the limit $\delta \rightarrow 0$.

$$\lim_{\delta \rightarrow 0} \langle \Delta | \hat{D}^{ab}(x, \delta) \hat{D}^{a'b'}(x', \delta) | f(G^\pm) \rangle = N^{-1}(\Delta) \sum_{\{\gamma\} \in \Delta} (-2)^{n(\{\gamma\})-1} (-1)^{c(\{\gamma\})} \times$$

$$\begin{aligned}
& \left[\int d^3y \frac{\delta f(G^\pm[\{\gamma\}])}{\delta G^\sigma[y, \{\gamma\}]} \lim_{\delta \rightarrow 0} \left(\frac{1}{4} \hat{D}^{ab}(x, \delta) \hat{D}^{a'b'}(x', \delta) G^\sigma[y, \{\gamma\}] \right) + \right. \\
& \int d^3y d^3y' \frac{\delta^2 f(G^\pm[\{\gamma\}])}{\delta G^\sigma[y, \{\gamma\}] \delta G^{\sigma'}[y', \{\gamma\}]} \lim_{\delta \rightarrow 0} \left(\frac{1}{2} \hat{D}^{ab}(x, \delta) G^\sigma[y, \{\gamma\}] \right) \times \\
& \left. \left(\frac{1}{2} \hat{D}^{a'b'}(x', \delta) G^{\sigma'}[y', \{\gamma\}] \right) \right]. \tag{36}
\end{aligned}$$

If $x' = x$, then we regularize the doubled operations at a single point by taking the limit $x' \rightarrow x$. This regularization means that we exclude the possibility that one \hat{D}^{ab} operator acts to the small loop contained in another $\hat{D}^{a'b'}$ operator. In other words, we allow the operator \hat{D}^{ab} act only to the spin-network under consideration. In the same way, we can apply the operator \hat{D}^{ab} more than twice to the state $|f(G^\pm)\rangle$ and evaluate it at the weave Δ and then take the limit $\delta \rightarrow 0$. They contain multiple operations of $\hat{D}^{ab}(x, \delta)$ to $G^\pm[y, \{\gamma\} \in \Delta]$.

In order to proceed the calculation of the action of \hat{D}^{ab} to $|f(G^\pm)\rangle$, compute the action of $\hat{D}^{ab}(x, \delta)$ to $G^\pm[y, \{\gamma\}]$ for $\{\gamma\} \in \Delta$ and then take the limit $\delta \rightarrow 0$.

$$\begin{aligned}
& \lim_{\delta \rightarrow 0} \frac{1}{2} \hat{D}^{ab}(x, \delta) G^\sigma[y, \{\gamma\} \in \Delta] := \lim_{\delta \rightarrow 0} \left[-\frac{1}{4} \frac{1}{\pi \delta^2} (-il_p^2) \right] \times \\
& \sum_{e \in \Delta} \sum_{s=1}^N \int dt l_e^b(t) \delta^3(\eta_{x,\delta}^a(s), l_e(t)) \sum_{i \in e} \left(G^\sigma[y, \{\gamma\}_i \eta_{x,\delta}] - G^\sigma[y, \{\gamma\}_i \eta_{x,\delta}^{-1}] \right) \\
& = \lim_{\delta \rightarrow 0} \left[-\frac{1}{4} \frac{1}{\pi \delta^2} (-il_p^2) \right] \sum_{e \in \Delta} \sum_{s=1}^N \rho \int dt l_e^b(t) \delta^3(\eta_{x,\delta}^a(s), l_e(t)) \times \\
& 2j_e \left[\sum_l \nu_l \frac{\dot{l}_e(t) \cdot \omega^{\sigma l}(l_e(t) - y)}{|\dot{l}_e(t) \cdot \omega^{\sigma l}(l_e(t) - y)|} \dot{\eta}_{x,\delta}^a(s) \cdot \omega^{\sigma l}(l_e(t) - y) + \mathcal{O}(\delta^2) \right] \\
& = -\frac{1}{4} (-il_p^2) \sum_l \nu_l \sum_{e \in \Delta} 2j_e \int dt l_e^b(t) \frac{\dot{l}_e(t) \cdot \omega^{\sigma l}(l_e(t) - y)}{|\dot{l}_e(t) \cdot \omega^{\sigma l}(l_e(t) - y)|} \omega_c^{\sigma l}(l_e(t) - y) \times \\
& \lim_{\delta \rightarrow 0} \frac{1}{\pi \delta^2} \oint ds (\dot{\eta}_{x,\delta}^a)^c(s) \delta^3(\eta_{x,\delta}(s), l_e(t)) \\
& = -\frac{1}{4} (-il_p^2) \sum_l \nu_l \sum_{e \in \Delta} 2j_e \int dt l_e^b(t) \frac{\dot{l}_e(t) \cdot \omega^{\sigma l}(l_e(t) - y)}{|\dot{l}_e(t) \cdot \omega^{\sigma l}(l_e(t) - y)|} \omega_c^{\sigma l}(l_e(t) - y) \times \\
& \epsilon^{acf} \partial_f^x \delta^3(x, l_e(t)). \tag{37}
\end{aligned}$$

Here, in the first step, the definition of the action of the operator $\hat{T}^a[\eta](\eta(s))$ was used and, in the second step, the property (21) of G^\pm was used on the limit of small δ . In the third step, the contributions from the weave and the small loop η were factored and, in the last step, the limit $\delta \rightarrow 0$ was taken. The line integral over the edges of Δ here does not have the regularized form since it does not make any confusion while the line integral over the loop η has the regularized form. Notice the transfer of the measure ρ from the function G^\pm to the integral of the operator. This process allows the non-vanishing contribution from the intersections of loops.

Compute the action of two $\hat{D}^{ab}(x, \delta)$ to $G^\pm[y, \{\gamma\}]$ and take the limit $\delta \rightarrow 0$.

$$\begin{aligned}
& \lim_{\delta \rightarrow 0} \frac{1}{4} \hat{D}^{ab}(x, \delta) \hat{D}^{a'b'}(x', \delta) G^\sigma[y, \{\gamma\}] := \lim_{\delta \rightarrow 0} \left[-\frac{1}{4} \frac{1}{\pi \delta^2} (-i l_p^2) \right]^2 \times \\
& \sum_{e \in \Delta} \sum_{e' \in \Delta} \sum_{s=1}^N \int dt l_e^b(t) \delta^3(\eta_{x, \delta}^a(s), l_e(t)) \sum_{s'=1}^N \int dt' l_{e'}^{b'}(t') \delta^3(\eta_{x', \delta}^{a'}(s'), l_{e'}(t')) \times \\
& \sum_{i \in e} \sum_{j \in e'} \left(G^\sigma[y, \{\gamma\}]_{ij} \eta_{x, \delta} \eta_{x', \delta} - G^\sigma[y, \{\gamma\}]_{ij} \eta_{x, \delta}^{-1} \eta_{x', \delta} - G^\sigma[y, \{\gamma\}]_{ij} \eta_{x, \delta} \eta_{x', \delta}^{-1} + \right. \\
& \left. G^\sigma[y, \{\gamma\}]_{ij} \eta_{x, \delta}^{-1} \eta_{x', \delta}^{-1} \right) = 0. \tag{38}
\end{aligned}$$

The vanishing result is due to the properties (22) and (23) of G^\pm . With the same reason, the action of more than two $\hat{D}^{ab}(x, \delta)$ to $G^\pm[y, \{\gamma\}]$ vanishes after taking the limit $\delta \rightarrow 0$.

Now, by using the action of $\hat{D}^{ab}[x, \delta]$ to $G^\pm[y, \{\gamma\}]$, apply the exponent of the operator $\hat{U}(\vec{\alpha})$ to $G^\pm[y, \{\gamma\}]$ for $\{\gamma\} \in \Delta$ and then take the limit $\delta \rightarrow 0$.

$$\begin{aligned}
& \frac{1}{2} \lim_{\delta \rightarrow 0} \left[-\frac{i}{2} \sum_i \int d^3x \epsilon_{abc} F_\xi^a[x, \alpha^i] \frac{\partial^b}{|\partial|^2} \hat{D}^{ci}(x, \delta) \right] G^\sigma[y, \{\gamma\} \in \Delta] \\
& = -\frac{i}{2} \sum_i \int d^3x \epsilon_{abc} F_\xi^a[x, \alpha^i] \frac{\partial_x^b}{|\partial_x|^2} \times \\
& \frac{1}{4} i l_p^2 \sum_l \nu_l \sum_{e \in \Delta} 2j_e \int dt l_e^i(t) \frac{\dot{l}_e(t) \cdot \omega^{\sigma l}(l_e(t) - y)}{|\dot{l}_e(t) \cdot \omega^{\sigma l}(l_e(t) - y)|} \omega_g^{\sigma l}(l_e(t) - y) \epsilon^{cgf} \partial_f^x \delta^3(x, l_e(t)) \\
& = -\frac{i}{2} \sum_i \int d^3x F_\xi^a[x, \alpha^i] \times \\
& \frac{1}{4} i l_p^2 \sum_l \nu_l \sum_{e \in \Delta} 2j_e \int dt l_e^i(t) \frac{\dot{l}_e(t) \cdot \omega^{\sigma l}(l_e(t) - y)}{|\dot{l}_e(t) \cdot \omega^{\sigma l}(l_e(t) - y)|} \omega_a^{\sigma l}(l_e(t) - y) \delta^3(x, l_e(t)). \tag{39}
\end{aligned}$$

Here, the first step was just the substitution of a previous result and the divergencelessness of $F_\xi^a[x, \alpha^i]$ was used in the second step.

Now, we note that the spins of all the edges of the weave have the same value j_Δ and use the following approximation formula.

$$\begin{aligned}
& l_p^2 \sum_{e \in \Delta} \sqrt{j_e(j_e + 1)} \int dt l_e^a(t) \frac{\dot{l}_e(t) \cdot v(l_e(t))}{|\dot{l}_e(t) \cdot v(l_e(t))|} v_b(l_e(t)) \\
& = \int d^3x \frac{v^a(x) v_b(x)}{\|v(x)\|} (1 + \mathcal{O}(l_p/L)), \tag{40}
\end{aligned}$$

where v_a is a covector slowly varying over the scale $L \gg l_p$ and $\|v\|$ is the norm of v and the integrals are performed over a space region of scale reasonably larger than l_p . This approximation formula can be proved in the same manner other approximation formulae were calculated [19, 20].

Here, $\omega_a^{\pm l}$ is a covector but not slowly varying over any particular scale and hence this approximation formula cannot be applied without further conditions. Since, in the domain states, $G^\pm[y, \{\gamma\}]$ always appears together with functions $f_{\sigma_1 \dots \sigma_n}(x_1 \dots x_n)$, which must be a slowly varying function over the scale $r := l_p/\varepsilon$ with respect to any of its arguments if they are successfully transmitted to the semi-classical theory, we restrict ourselves to the case that $\omega_a^{\pm l}(l_e(t) - y)$ is smeared out over the scale r with respect to y . The integrals are performed over regions of scale $\xi \gg l_p$. Then

$$\begin{aligned}
& \frac{1}{2} \lim_{\delta \rightarrow 0} \left[-\frac{i}{2} \sum_i \int d^3 x \epsilon_{abc} F_\xi^a[x, \alpha^i] \frac{\partial^b}{|\partial|^2} \hat{D}^{ci}(x, \delta) \right] G^\sigma[y, \{\gamma\} \in \Delta] \\
&= -\frac{i}{2} \sum_i \int d^3 x F_\xi^a[x, \alpha^i] \frac{2j_\Delta}{\sqrt{j_\Delta(j_\Delta + 1)}} \frac{i}{2} \delta^{id} \times \\
& \int d^3 z \sum_l \frac{1}{2} \nu_l \frac{\omega_d^{\sigma l}(z - y) \omega_a^{\sigma l}(z - y)}{\|\omega^{\sigma l}(z - y)\|} (1 + \mathcal{O}(\varepsilon)) \delta^3(x, z) \\
&= \frac{1}{4} \sum_i \int d^3 x F_\xi^a[x, \alpha^i] \frac{2j_\Delta}{\sqrt{j_\Delta(j_\Delta + 1)}} \delta^{id} \sum_l \frac{1}{2} \nu_l \frac{\omega_d^{\sigma l}(x - y) \omega_a^{\sigma l}(x - y)}{\|\omega^{\sigma l}(x - y)\|} (1 + \mathcal{O}(\varepsilon)) \\
&= \frac{1}{4} \frac{2j_\Delta}{\sqrt{j_\Delta(j_\Delta + 1)}} \sum_i \int d^3 x F_\xi^a[x, \alpha^i] (1 + \mathcal{O}(\varepsilon)) \times \\
& (2\pi)^{-3} \int d^3 k |k|^{-2} m^i(k) m_a(k) e^{\sigma i k \cdot (x - y)} \\
&= \frac{1}{4} \frac{2j_\Delta}{\sqrt{j_\Delta(j_\Delta + 1)}} \sum_i (2\pi)^{-3/2} \int d^3 k |k|^{-2} m^i(k) m_a(k) F_\xi^a[-\sigma k, \alpha^i] e^{-\sigma i k \cdot y} (1 + \mathcal{O}(\varepsilon)) \\
&= \frac{1}{4} \frac{2j_\Delta}{\sqrt{j_\Delta(j_\Delta + 1)}} (2\pi)^{-3/2} \int d^3 k |k|^{-2} F_\xi^\sigma[k, \vec{\alpha}] e^{i k \cdot y} (1 + \mathcal{O}(\varepsilon)) \\
&= \frac{1}{4} \frac{2j_\Delta}{\sqrt{j_\Delta(j_\Delta + 1)}} F_\xi^\sigma[y, \vec{\alpha}] (1 + \mathcal{O}(\varepsilon)). \tag{41}
\end{aligned}$$

Here, in the first step, the approximation formula was used to replace the sum of the line integrals along the edges of the weave by an integral over space z and the integration was done in the second step. In the third step, the definition of $\omega_a^{\pm l}$ in terms of the polarization vectors was used and, subsequently, Fourier transformations have been repeated to find the final line.

Before evaluating the transformation, apply the exponent of the operator $\hat{U}[\vec{\alpha}]$ to $|f(G^\pm)\rangle$ and evaluate it at the weave Δ and take the limit $\delta \rightarrow 0$.

$$\begin{aligned}
& \lim_{\delta \rightarrow 0} \langle \Delta | \left[-\frac{i}{2} \sum_i \int d^3 x \epsilon_{abc} F_\xi^a[x, \alpha^i] \frac{\partial^b}{|\partial|^2} \hat{D}^{ci}(x, \delta) \right] |f(G^\pm)\rangle = \\
& N^{-1}(\Delta) \sum_{\{\gamma\} \in \Delta} (-2)^{n(\{\gamma\})-1} (-1)^{c(\{\gamma\})} \int d^3 y \frac{\delta f(G^\pm[\{\gamma\}])}{\delta G^\sigma[y, \{\gamma\}]} \lambda_1(\Delta, \omega^\pm) F_\xi^\sigma[y, \vec{\alpha}], \tag{42}
\end{aligned}$$

where

$$\lambda_1(\Delta, \omega^\pm) := \frac{1}{4} \frac{2j_\Delta}{\sqrt{j_\Delta(j_\Delta + 1)}} (1 + \mathcal{O}(\varepsilon)), \quad (43)$$

which is a number of order 1. $\mathcal{O}(\varepsilon)$ depends on $\omega_a^{\pm l}$ and Δ .

In the same way, apply the exponent of the operator $\hat{U}[\vec{\alpha}]$ twice to $|f(G^\pm)\rangle$ and evaluate it at the weave Δ and take the limit $\delta \rightarrow 0$.

$$\begin{aligned} \lim_{\delta \rightarrow 0} \langle \Delta | \frac{1}{2} \left[-\frac{i}{2} \sum_i \int d^3x \epsilon_{abc} F_\xi^a[x, \alpha^i] \frac{\partial^b}{|\partial|^2} \hat{D}^{ci}(x, \delta) \right]^2 |f(G^\pm)\rangle = \\ N^{-1}(\Delta) \sum_{\{\gamma\} \in \Delta} (-2)^{n(\{\gamma\})-1} (-1)^{c(\{\gamma\})} \times \\ \frac{1}{2} \int d^3y d^3y' \frac{\delta^2 f(G^\pm[\{\gamma\}])}{\delta G^\sigma[y, \{\gamma\}] \delta G^{\sigma'}[y', \{\gamma\}]} \lambda_1 F_\xi^\sigma[y, \vec{\alpha}] \lambda_1 F_\xi^{\sigma'}[y', \vec{\alpha}]. \end{aligned} \quad (44)$$

In general, the n -th power of the exponent of the operator $\hat{U}(\vec{\alpha})$ applying to $|f(G^\pm)\rangle$ at the weave Δ with the limit $\delta \rightarrow 0$ can be computed as follows.

$$\begin{aligned} \lim_{\delta \rightarrow 0} \langle \Delta | \left[-\frac{i}{2} \sum_i \int d^3x \epsilon_{abc} F_\xi^a[x, \alpha^i] \frac{\partial^b}{|\partial|^2} \hat{D}^{ci}(x, \delta) \right]^n |f(G^\pm)\rangle = \\ N^{-1}(\Delta) \sum_{\{\gamma\} \in \Delta} (-2)^{n(\{\gamma\})-1} (-1)^{c(\{\gamma\})} \times \\ \int d^3y_1 \cdots d^3y_n \frac{\delta^n f(G^\pm[\{\gamma\}])}{\delta G^{\sigma_1}[y_1, \{\gamma\}] \cdots \delta G^{\sigma_n}[y_n, \{\gamma\}]} \lambda_1 F_\xi^{\sigma_1}[y_1, \vec{\alpha}] \cdots \lambda_1 F_\xi^{\sigma_n}[y_n, \vec{\alpha}]. \end{aligned} \quad (45)$$

From these results, it is easy to evaluate the transformation to the state $|f(G^\pm)\rangle$ at the weave Δ .

$$\begin{aligned} \langle \Delta | \hat{U}(\vec{\alpha}) |f(G^\pm)\rangle = N^{-1}(\Delta) \sum_{\{\gamma\} \in \Delta} (-2)^{n(\{\gamma\})-1} (-1)^{c(\{\gamma\})} \times \\ \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3y_1 \cdots d^3y_n \frac{\delta^n f(G^\pm[\{\gamma\}])}{\delta G^{\sigma_1}[y_1, \{\gamma\}] \cdots \delta G^{\sigma_n}[y_n, \{\gamma\}]} \lambda_1 F_\xi^{\sigma_1}[y_1, \vec{\alpha}] \cdots \lambda_1 F_\xi^{\sigma_n}[y_n, \vec{\alpha}] \\ = C(\Delta) f(\lambda_0(\Delta, \omega^\pm) + \lambda_1(\Delta, \omega^\pm) F_\xi^\pm[\vec{\alpha}]). \end{aligned} \quad (46)$$

Here $C(\Delta)$ is a multiplicative constant depending on the weave defined by

$$C(\Delta) := N^{-1}(\Delta) \sum_{\{\gamma\} \in \Delta} (-2)^{n(\{\gamma\})-1} (-1)^{c(\{\gamma\})}, \quad (47)$$

and $f(\lambda_0 + \lambda_1 F_\xi^\pm[\vec{\alpha}])$ is $f(G^\pm[\{\gamma\}])$ with $G^\pm[x, \{\gamma\}]$ replaced by $\lambda_0 + \lambda_1 F_\xi^\pm[x, \vec{\alpha}]$. λ_0 is defined for $\{\gamma\} \in \Delta$ by

$$\lambda_0(\Delta, \omega^\pm) := G^\pm[x, \{\gamma\} \in \Delta] \xrightarrow{\rho \rightarrow 0, N \rightarrow \infty} \longrightarrow$$

$$\begin{aligned}
& \sum_{e \in \Delta} 2j_e \sum_l \frac{1}{2} \nu_l \int dt |l_e^a(t) \omega_a^{\pm l}(l_e(t) - x)| \\
&= \frac{2j_\Delta}{\sqrt{j_\Delta(j_\Delta + 1)}} \sum_l \frac{1}{2} \nu_l \sum_{e \in \Delta} \sqrt{j_e(j_e + 1)} \int dt |l_e^a(t) \omega_a^{\pm l}(l_e(t) - x)| \\
&= l_p^{-2} \frac{2j_\Delta}{\sqrt{j_\Delta(j_\Delta + 1)}} \sum_l \frac{1}{2} \nu_l \int dz \|\omega^{\pm l}(z - x)\| (1 + \mathcal{O}(\varepsilon)) \\
&= l_p^{-2} \frac{2j_\Delta}{\sqrt{j_\Delta(j_\Delta + 1)}} \sum_l \frac{1}{2} \nu_l \int dz \|\omega^{\pm l}(z - x)\| \mathcal{O}(\varepsilon), \tag{48}
\end{aligned}$$

where the leading term has vanished because of $\sum_l \frac{1}{2} \nu_l \|\omega^{\pm l}\| = 0$ at each space point while the other term has survived since the error $\mathcal{O}(\varepsilon)$ depends on space point implicitly through $\omega_a^{\pm l}$ and Δ . In the computation of λ_0 , we have used another approximation formula as follows.

$$l_p^2 \sum_{e \in \Delta} \sqrt{j_e(j_e + 1)} \int dt |l_e^a(t) v_a(l_e(t))| = \int d^3x \|v(x)\| (1 + \mathcal{O}(l_p/L)). \tag{49}$$

Again, as in the other approximation formula, v_a is a covector slowly varying over the scale $L \gg l_p$ and $\|v\|$ is the norm of v . The integrals are performed over a space region of scale reasonably larger than l_p . This approximation formula is already known [19, 20], or more precisely, the weave is defined such a way that this approximation formula holds.

The transformation \mathcal{N} transforms an analytic function of G^\pm to the same analytic function with G^\pm replaced by $\lambda_0 + \lambda_1 F_\xi^\pm$ provided that the coefficient functions such as $f_{\sigma_1 \dots \sigma_n}(x_n \dots x_n)$ are slowly varying functions over the scale r with respect to any of their arguments. In particular, G^\pm itself is transformed to $\lambda_0 + \lambda_1 F_\xi^\pm$, if it is smeared against a slowly varying function over the scale r , as follows.

$$\lambda_0(\Delta, \omega) + \lambda_1(\Delta, \omega) F_\xi^\pm[x, \vec{\alpha}] = \langle \Delta | \hat{U}(\vec{\alpha}) | G^\pm(x) \rangle. \tag{50}$$

Here λ_1 and λ_0 are of order 1 and ε respectively and depend on the weave and $\omega^{\pm l}$. These terms are Planck scale corrections coming from the difference between the discrete structure of the weave and the flat space structure assumed at all the scales in the semi-classical theory.

Now, what is of interest is that the Planck scale correction terms appear always with F^\pm . One is multiplicative to and the other additive to F^\pm . We notice that in the semi-classical theory the Planck length constant enters the theory always multiplicative to F^\pm and a constant field additive to F^\pm defines a unitarily equivalent theory. Therefore, the multiplicative correction can be absorbed to the original Planck length constant l_p to redefine a “coarse-grained” constant \bar{l}_p . Then the additive correction term can be eliminated by a unitary transformation of the variables of the semi-classical theory.

After all, the transformation \mathcal{N} transforms an analytic function of G^\pm to the same analytic function with G^\pm replaced by F_ξ^\pm and with l_p replaced by a “coarse-grained”

constant \bar{l}_p provided that the coefficient functions contained are slowly varying functions over the scale r . Here the parameter ξ is so small compared to the scale of graviton fluctuations r , we disregard the difference between F^\pm and F_ξ^\pm on the physical ground and simply consider F_ξ^\pm as a regularization form of F^\pm . Accordingly, the multiplication of and the derivative with respect to G^\pm respectively correspond through the transformation to the multiplication of and the derivative with respect to F_ξ^\pm . Since the multiplication of and the derivative with respect to F^\pm are the basic operations of the semi-classical theory, the operators \hat{H}^\pm and \hat{D}^{ab} constructed in section 2 provide the basic operations on the domain states for gravitons. \hat{H}^\pm and suitable components of \hat{D}^{ab} define respectively the multiplication of and the derivative with respect to G^\pm in the non-perturbative theory in the same sense that \hat{h}^\pm and \hat{B}^\pm define respectively the multiplication of and the derivative with respect to F^\pm in the semi-classical theory provided that $f_r(x, y)$ in \hat{H}^\pm is identified to $g_r(x - y)$ in \hat{h}^\pm .

5 Comparison with the previous work

In this section, we compare the present work with the previous work [5, 6]. In particular, we realize how the transformation \mathcal{N} constructed in the present work improves the interpretation of the graviton states in the non-perturbative theory. The interpretation was partially made in the previous work.

5.1 The previous work: the transformation \mathcal{M}

In the previous work, a transformation denoted by \mathcal{M} was constructed. It transforms a family of states, the prototype of the family of states presented in the present work, in the non-perturbative loop representation space to states in the semi-classical loop representation space. However, \mathcal{M} contains an error term due to its definition and its use is limited to transformations of states and operators of the first order magnitude in the fluctuations of gravitational field in terms of suitable norm. From this limitation the functions $G^\pm[x, \gamma]$ and some operator \hat{A} were found to be related respectively to the basis states $F^\pm[x, \vec{\alpha}]$ (the Fourier transform of the symmetric traceless transverse components of $F^a[k, \alpha^i]$) and the annihilation operator \hat{a} (a linear combination of the basic canonical variables) of the semi-classical theory, that is

$$F^\pm = \mathcal{M}G^\pm + \mathcal{O}(\varepsilon^2) \quad (51)$$

$$\hat{a}\mathcal{M} = \mathcal{M}\hat{A} + \mathcal{O}(\varepsilon^2). \quad (52)$$

Here ε is the order of the fluctuations of gravitational field.

By interpreting G^\pm and \hat{A} as basis states and annihilation operator in the non-perturbative loop representation space respectively, the graviton states were constructed INDIRECTLY. For example, The ground state was found by requiring that it is annihilated by the annihilation operator \hat{A} up to an error of order ε^2 and it is a function of G^\pm .

5.2 The present work: the transformation \mathcal{N}

In the present work, a transformation denoted by \mathcal{N} was constructed. It transforms a family of states containing a function of G^\pm in the non-perturbative loop representation space to the state with the form of the same function of F_ξ^\pm in the semi-classical loop representation space WITHOUT an error term, that is

$$f(\lambda_1 F_\xi^\pm + \lambda_0) = \mathcal{N}f(G^\pm). \quad (53)$$

The constants λ_0 and λ_1 have emerged as “deviations” of the weave structure from flat space. By redefining the Planck length constant in the semi-classical theory and performing a unitary transformation of the canonical variables to eliminate λ_1 and λ_0 respectively, we found

$$f(F_\xi^\pm) = \mathcal{N}f(G^\pm). \quad (54)$$

From this we can immediately find the graviton states in the non-perturbative theory as before but DIRECTLY without solving any requirement since they are just special cases of functions of G^\pm .

An important difference is the following. The transformation \mathcal{N} does not contain an error term. Therefore, the definition of the graviton states in the non-perturbative theory is “exact” although they are approximate states in the sense that they have nothing to do with the exact constraints of the non-perturbative theory. The transformation \mathcal{M} allows the interpretation that these states represent free gravitons up to an error of order ε^2 because of the presence of an error term in it. This error term spoils the interpretation when the self-interaction of gravitons is introduced possibly perturbatively or non-perturbatively. The transformation \mathcal{N} keeps the interpretation valid even after introducing the self-interaction because of the absence of error term.

6 Conclusions

We have constructed, in the non-perturbative loop representation space, what we call basis states for gravitons, denoted by G^\pm , in terms of which a family of states are defined. We have showed that there exists a transformation which transforms the family of states to the domain states of the semi-classical loop representation space. Given a state made of an analytic function of G^\pm with a parameter l_p (the Planck length constant), denoted by $f(G^\pm)$, the transformation transforms it to the same analytic function with G^\pm and l_p replaced by F_ξ^\pm (the basis states of the semi-classical loop representation space) and \bar{l}_p (a “coars-grained” constant) respectively up to an overall multiplicative constant.

Therefore, all the domain states of the semi-classical loop representation space can be recovered, through the transformation \mathcal{N} , from the family of states in the non-perturbative loop representation space. In particular, the states corresponding to the graviton states of the semi-classical theory can be found easily since they are special cases of the family of states. They are the graviton states in the non-perturbative

theory. This relation of the two theories is “exact” in the sense that the inclusion of self-interaction of gravitons does not make the interpretation of these graviton states obscure.

Although to prove a mathematical exactness such as isomorphism of some sectors of the two theories is not clear, the existence of this “exact” relation of the two theories supports the potential ability of the non-perturbative loop representation quantum gravity to address the physics of gravitons, namely quanta for large scale small fluctuations of gravitational field from the flat background spacetime.

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